Statistical theory of ideal quantum measurement processes

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To study ideal measurement processes involving a tested system S coupled to an apparatus A, we rely on a minimalist, statistical formulation of quantum mechanics, where states encode properties of ensembles. The required final state of S+A is shown to have a Gibbsian thermodynamic equilibrium form, not only for a large ensemble of runs, but also for arbitrary subensembles. This outcome is justified dynamically in quantum statistical mechanics as result of relaxation for models with suitably chosen interactions within A. The known quantum ambiguity which hinders the identification of physical subensembles is overcome owing to a specific relaxation process. The structure of the states describing the subsets of runs provides a frequency interpretation of Born's rule and explains the uniqueness of the result for each individual run, the so-called measurement problem.

One of the main foundational challenges of quantum theory is the so-called measurement problem: Why does each individual run of an ideal measurement yield a well-defined outcome, in spite of the existence of quantum coherences? Does measurement theory require a specific principle or interpretation of quantum mechanics? Already raised by the founding fathers, this crucial question has witnessed a revival [1–4]. Many authors explore models, others propose alternative interpretations of quantum mechanics or go beyond it. We shall, however, approach the problem by just treating the tested system S coupled to the measurement apparatus A as a single isolated quantum system S+A, and analysing the dynamics of the measurement process in a straightforward way. A key point is the *macroscopic size* of the apparatus, which forces us to rely on non-equilibrium quantum statistical mechanics. Moreover, being irreducibly probabilistic, quantum physics does not describe individual objects; we must deal with *statistical ensembles*, and attempt to infer the properties of a single run of the measurement from those of the subensembles of runs to which it may belong.

Technically, we have to solve the equations of motion for the joint density operator $\hat{\mathcal{D}}(t)$ of S+A. We denote by $\hat{s} = \sum_i s_i \hat{\Pi}_i$ the observable of S to be measured, which has the eigenvalues s_i and the associated eigenprojections $\hat{\Pi}_i$, and by $\hat{r}(t) = \mathrm{tr}_A \hat{\mathcal{D}}(t)$ and $\hat{\mathcal{R}}(t) = \mathrm{tr}_S \hat{\mathcal{D}}(t)$ the marginal density operators of S and A, respectively. At the initial time $t = 0$, S and A are uncorrelated, S lies in some state $\hat{r}(0)$ and A in a metastable state $\hat{\mathcal{R}}(0)$, so that $\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$. Triggered by its coupling with S, A may switch towards one or another among its stable states $\hat{\mathcal{R}}_i$, which should have equal entropies and energies so as to avoid bias in the measurement. These states can be distinguished from one another through observation or registration of the pointer variable A_i, identified with the expectation value $\mathrm{tr}_A \hat{R}_i \hat{A}$ of some collective observable $\hat{A}$ of A. The final indication A_i of the pointer must be fully correlated with the eigenvalue s_i of the tested observable $\hat{s}$ and with the production of the final state $\hat{r}_i$ for S. We wish therefore to show that, *for each run of the measurement*, S+A reaches at some final time t_f one among the states [5]

$$\hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i, \qquad p_i \hat{r}_i = \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i, \qquad (1)$$

with Born's probability $p_i = \langle \hat{\Pi}_i \rangle = \mathrm{tr}_S \hat{\Pi}_i \hat{r}(0)$. Parts of this task have been achieved for models reviewed in [1–4], while a detailed solution of the Curie–Weiss model [1, 6]. illustrates its steps. We focus below on some key ideas which apply to *arbitrary* ideal measurements.

1. Statistical formulation of the principles of quantum mechanics. We tackle the measurement problem within the statistical formulation of quantum mechanics, also called "statistical interpretation" or "*ensemble* interpretation". It is a *minimalist*, formal description and does not prejudge an interpretation of the quantum oddities [7]. Physical interpretations should emerge at the macroscopic scale, in experimental contexts, as will be discussed in § 5 for ideal measurements.

Physical quantities pertaining to an ensemble of systems are described as observables represented by Hermitean matrices in a Hilbert space. Observables are comparable to random variables, but their randomness is unescapable, since it arises from their non-commutation, not solely from a lack of knowledge or of control.

A quantum state, whether pure or not, encompasses the probabilistic predictions that one may make about the various observables. It is characterised by a correspondence between observables $\hat{O}$ and real numbers, implemented as $\hat{O} \mapsto \langle \hat{O} \rangle = \mathrm{tr} \hat{\mathcal{D}} \hat{O}$ by means of a Hermitean, normalised and non-negative density operator $\hat{\mathcal{D}}$. Such a definition looks analogous to that of a state in classical statistical mechanics, encoded in a density in phase space. However, "quantum expectation values" $\langle \hat{O} \rangle$, "quantum probabilities" such as $\langle \hat{\Pi}_i \rangle$ and "quantum correlations" such as $\langle \hat{s} \hat{A} \rangle$ present a priori only a formal similarity with standard expectation values, probabilities and correlations; they express not only some ignorance as in classical physics, but also the inadequacy to represent physical quantities by *c*-numbers.

As a standard probability distribution, a state of a system gathers information about it, and hence refers, implicitly or not, to a statistical ensemble $\mathcal{E}$. So a state does not "belong to a system", it is not an intrinsic property: we should regard an individual system as an element of a large, real or virtual, statistical ensemble $\mathcal{E}$ of systems produced under the same conditions. As in ordinary probability theory, we may update our information by selecting some subensemble $\mathcal{E}_{\mathrm{sub}}$ to which the considered system belongs, and thus assign to it a new state.

2. System plus apparatus in thermodynamic equilibrium. We suppose that the compound system S+A is isolated, including in A a thermal bath or an environment if present. As usual for macroscopic systems, isolation of S+A is implemented by specifying the average energy. The properties of S+A are governed by the Hamiltonian $\hat{H} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{SA}}$, which must have specific features. The part $\hat{H}_{\mathrm{A}}$ associated with the macroscopic apparatus should produce several thermodynamic equilibrium states $\hat{\mathcal{R}}_i$ if A is decoupled from S. A typical example is provided by spontaneously broken discrete invariance, the pointer variable A_i being the order parameter.

As we wish to deal with ideal, non-demolishing measurements, the process should perturb S as little as possible: any observable compatible with $\hat{s}$, i.e., commuting with its eigenprojectors $\hat{\Pi}_i$, should remain unaffected. The conservation of all these observables is expressed by the fact that $\hat{H}$ depends on S only through the projectors $\hat{\Pi}_i$. In particular the coupling must have the form

$$\hat{H}_{\mathrm{SA}} = \sum_i \hat{\Pi}_i \otimes \hat{h}_i, \qquad (2)$$

where $\hat{h}_i$ are operators of A. Moreover, if $\hat{s}$ takes the value s_i, that is, $\hat{\Pi}_i$ the value 1, A should lie near its stable state $\hat{\mathcal{R}}_i$, the pointer variable being close to A_i. This can be achieved if each $\hat{h}_i$ depends mainly on the pointer observable $\hat{A}$ and behaves as a source that breaks explicitly the invariance by energetically favouring A_i.

In this section, we disregard the dynamics of the measurement process and determine the general form $\hat{\mathcal{D}}_{\mathrm{eq}}$ for the thermodynamic equilibrium states of the compound system S+A. To this end we employ the maximum entropy method, a basic criterion for assigning a state to a system in equilibrium [8]. We thus maximise von Neumann's entropy $-\mathrm{tr}\hat{\mathcal{D}} \ln \hat{\mathcal{D}}$ under constraints accounting for the conserved quantities, which are here the energy $\langle \hat{H} \rangle$ and the expectation values of the set of all observables of S commuting with the projections $\hat{\Pi}_i$.

Associating Lagrange multipliers with all such expectation values, we find for the equilibrium states a Gibbsian form where the exponent involves an arbitrary linear combination of these conserved observables, i.e., $\hat{\mathcal{D}}_{\mathrm{eq}} \propto \exp(-\beta \hat{H} + \sum_i \hat{y}_i)$ where $\hat{y}_i$ is any operator of S acting inside the diagonal block i (so that $\hat{y}_i = \hat{\Pi}_i \hat{y}_i \hat{\Pi}_i$). Noting that the exponent, which commutes with the projections

$\hat{\Pi}_i$, has a block diagonal structure, we find for these equilibrium states the structure

$$\hat{\mathcal{D}}_{\mathrm{eq}} = \sum_i q_i \hat{x}_i \otimes \hat{\mathcal{R}}_i^h, \qquad \sum_i q_i = 1. \qquad (3)$$

Each factor $q_i \hat{x}_i$, which arises from $\exp(-\beta \hat{H}_{\mathrm{S}} + \hat{y}_i)$, is an arbitrary block diagonal operator of S, where $\hat{x}_i = \hat{\Pi}_i \hat{x}_i \hat{\Pi}_i$, $\mathrm{tr}_{\mathrm{S}} \hat{x}_i = 1$ can be interpreted as a density operator of S. (If the eigenvalue s_i is non degenerate, one just has $\hat{x}_i = \hat{\Pi}_i$.) Furthermore, each factor $\hat{\mathcal{R}}_i^h \propto \exp[-\beta(\hat{H}_{\mathrm{A}} + \hat{h}_i)]$ in (3) can be interpreted as a canonical equilibrium density operator in the space of A. Here the invariance is *explicitly broken* by adding to the Hamiltonian of A the source term $\hat{h}_i$ that arises from (2). This term should be sufficiently large so that $\hat{\mathcal{R}}_i^h$ contains no admixture of $i' \neq i$ for a macroscopic apparatus (otherwise different s_i's are not properly distinguished). It should not be too large though, so that $\hat{\mathcal{R}}_i^h$ relaxes to $\hat{\mathcal{R}}_i$ after $\hat{H}_{\mathrm{SA}}$ is switched off. Thermodynamic equilibrium (3) entails a complete correlation between the eigenvalue s_i of $\hat{s}$ and the value A_i of the pointer variable.

Returning to a measurement process described by the dynamics of a statistical ensemble, consider a large set $\mathcal{E}$ of runs for which S+A lies initially in the state $\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$. The final state for the full ensemble $\mathcal{E}$ is expected from (1) to be

$$\hat{\mathcal{D}}(t_{\mathrm{f}}) = \sum_i p_i \hat{\mathcal{D}}_i = \sum_i p_i \hat{r}_i \otimes \hat{\mathcal{R}}_i. \qquad (4)$$

We can identify it with a thermodynamic equilibrium state (3) with $q_i = p_i$ and $\hat{x}_i = \hat{r}_i$, for which relaxation of $\hat{\mathcal{R}}_i^h \to \hat{\mathcal{R}}_i$ has taken place after switching off $\hat{H}_{\mathrm{SA}}$. The free parameters of $\hat{\mathcal{D}}_{\mathrm{eq}}$ are determined by the initial state $\hat{\mathcal{D}}(0)$ and the conservation laws as $q_i \hat{x}_i = \hat{\Pi}_i \hat{r}(0) \hat{\Pi}_i$.

The state $\hat{\mathcal{D}}_i$ defined in (1), expected to describe the subensemble $\mathcal{E}_i$ obtained by selecting within $\mathcal{E}$ the runs tagged by the value A_i of the pointer, is also identified with an equilibrium state for which all q_j with $j \neq i$ vanish. For an *arbitrary subset* $\mathcal{E}_{\mathrm{sub}}$ of runs having yielded a proportion q_i of individual runs with outcomes A_i, the expected final state

$$\hat{\mathcal{D}}_{\mathrm{sub}}(t_{\mathrm{f}}) = \sum_i q_i \hat{\mathcal{D}}_i \qquad (5)$$

is again a thermodynamic equilibrium state (3) with $\hat{x}_i = \hat{r}_i$ (see § 4). Thus, a dynamic confirmation of (4) and (5) amounts to prove that S+A relaxes from $\hat{\mathcal{D}}(0)$ to Gibbsian equilibrium, for $\mathcal{E}$ and for its arbitrary subensembles $\mathcal{E}_{\mathrm{sub}}$, in an evolution governed by $\hat{H}$. The proof must rely on quantum statistical dynamics and on adequate properties of A. It should also provide the characteristic time scales of the process. We sketch its main steps in sections 3 and 4.

3. Dynamics of S+A for the full set of runs. We first outline some general features of the dynamics of the ensemble $\mathcal{E}$ of coupled systems S and A. The evolution of $\hat{\mathcal{D}}(t)$, though governed by the Liouville–von Neumann equation $i\hbar\,\mathrm{d}\hat{\mathcal{D}}/\mathrm{d}t = [\hat{H},\hat{\mathcal{D}}]$, is irreversible. This implies, as usual in statistical mechanics, the use of approximations, which should be justified through a suitable choice of parameters in $\hat{H}_A$ and $\hat{H}_{SA}$. For a large apparatus, we are permitted to disregard correlations between a macroscopic number of degrees of freedom: In agreement with the definition of a state, we may drop information about unattainable observables of A. We are also allowed to forget about recurrences that would occur after unreasonably large times. A dynamical explanation of quantum measurements thus requires physically justified approximations, based on a suitable choice of parameters in $\hat{H}_A$ and $\hat{H}_{SA}$, in the same way as the explanation of irreversibility in classical statistical mechanics [1].

Taking into account the properties of $\hat{H}$, in particular the form (2) of $\hat{H}_{SA}$ and the property $[\hat{H}_S,\hat{r}(0)] = 0$, which ensures that the state $\hat{r}(t)$ of S is perturbed only by the interaction $\hat{H}_{SA}$, it is easy to check that $\hat{\mathcal{D}}(t)$ can be parameterised as

$$\hat{\mathcal{D}}(t) = \sum_{i,j} \hat{\Pi}_i\hat{r}(0)\hat{\Pi}_j \otimes \hat{\mathcal{R}}_{ij}(t) \qquad (6)$$

in terms of a set $\hat{\mathcal{R}}_{ij}(t) = \hat{\mathcal{R}}_{ji}^{\dagger}(t)$ of operators in the Hilbert space of A, to be determined by the equations

$$i\hbar\frac{\mathrm{d}\hat{\mathcal{R}}_{ij}(t)}{\mathrm{d}t} = (\hat{H}_A + \hat{h}_i)\hat{\mathcal{R}}_{ij}(t) - \hat{\mathcal{R}}_{ij}(t)(\hat{H}_A + \hat{h}_j), \quad (7)$$

with the initial conditions $\hat{\mathcal{R}}_{ij}(0) = \hat{\mathcal{R}}(0)$. The dynamics thus involves *solely the apparatus*, irrespective of the tested system, a specific property of ideal measurements.

To derive dynamically (4) from (7), one has to study two independent processes, characterised by different time scales. For $i \neq j$, the coherent contributions $\hat{\mathcal{R}}_{ij}(t)$ should decay so as to eventually vanish. For $i = j$, $\hat{\mathcal{R}}_{ii}(t)$ should relax to the equilibrium state $\hat{\mathcal{R}}_i^h$ in the presence of the source $\hat{h}_i$ (which requires this source to be sufficiently large), and then to $\hat{\mathcal{R}}_i$ after $\hat{H}_{SA}$ is switched off; the required correlation between s_i and A_i is thereby established. Both processes are essential. They resort to standard techniques of non-equilibrium statistical mechanics, which have been applied to many models [1–4]. It will therefore be sufficient to acknowledge here this possibility of confirming the thermodynamic result (4) through microscopic dynamics. However, the understanding of measurements requires further steps.

4. Final states for arbitrary subensembles. Indeed, once the required expression (4) of the final state of S+A for the full ensemble $\mathcal{E}$ of runs has been derived, the task is not yet achieved. We also need to prove that all the states $\hat{\mathcal{D}}_{sub}(t)$ associated with arbitrary subsets $\mathcal{E}_{sub}$ of runs reach an equilibrium form (5). *All possible* subsets must be considered; those obtained by extracting at random runs from the full set $\mathcal{E}$ are useless for our purpose, since they are described by $\hat{\mathcal{D}}(t)$ itself. To study a subensemble described by $\hat{\mathcal{D}}_{sub}(t)$, we can rely on the dynamical equations (7), but we now lack information about *initial conditions.* All we know is that, at a time t_{split} taken as initial time for $\hat{\mathcal{D}}_{sub}$, $\hat{\mathcal{D}}$ can be split as

$$\hat{\mathcal{D}}(t_{split}) = k\hat{\mathcal{D}}_{sub}(t_{split}) + (1-k)\hat{\mathcal{D}}_{Csub}(t_{split}), \quad (8)$$

where $0 < k < 1$ denotes the proportion of runs in $\mathcal{E}_{sub}$ and where the state $\hat{\mathcal{D}}_{Csub}$ describes the complement of $\mathcal{E}_{sub}$. However, if we conversely decompose the given state $\hat{\mathcal{D}}$ as in (8), nothing tells us that there exists a subset of runs extracted from the full ensemble $\mathcal{E}$ which would be described by the component $\hat{\mathcal{D}}_{sub}$. This difficulty arises from a *quantum ambiguity*, to wit, a mixed state possesses many different *physically incompatible decompositions*. As defined in section 1, a state characterises a correspondence $\hat{O} \mapsto \langle\hat{O}\rangle = \mathrm{tr}\hat{\mathcal{D}}\hat{O}$. In classical probability theory, specifying likewise the expectation value of any random variable implies the existence of probabilities for individual events, and hence determines the probability distributions for the subensembles of events. On the contrary, in quantum theory, it is fallacious to believe that a term $\hat{\mathcal{D}}_{sub}$ issued from some mathematical splitting (8) of $\hat{\mathcal{D}}$ has any physical relevance in terms of a real subensemble. As an illustration, consider an unpolarised ensemble of spins; its density operator equals both $\frac{1}{2}(|z\rangle\langle z| + |-z\rangle\langle -z|)$ and $\frac{1}{2}(|x\rangle\langle x| + |-x\rangle\langle -x|)$ where $|z\rangle$ denotes a pure state polarised in the z-direction. If these two decompositions were meaningful, there would exist subensembles of spins polarised simultaneously in two orthogonal directions! Likewise, as any mixed state, $\hat{\mathcal{D}}$ possesses physically incompatible decompositions, so that we cannot infer from its sole knowledge whether a term $\hat{\mathcal{D}}_{sub}$ of (8) is or not associated with one among the real subsets of runs $\mathcal{E}_{sub}$ in which we are interested.

Our strategy will therefore rely on the consideration of *any mathematically allowed* decomposition (8) of $\hat{\mathcal{D}}(t_{split})$, the first element of which being taken as initial state $\hat{\mathcal{D}}_{sub}(t_{split})$. Although nothing ensures that the operators $\hat{\mathcal{D}}_{sub}(t)$ thus defined are physically meaningful (and in most cases they will not), we are ascertained that their class will *include the physical states* describing all real subsets of runs that appear in practice.

By studying the dynamics of $\hat{\mathcal{D}}_{sub}(t)$ for $t > t_{sub}$, we wish to prove that any admissible candidate $\hat{\mathcal{D}}_{sub}(t)$ for a physical state ends up in the expected form $\hat{\mathcal{D}}_{sub}(t_f) = \sum_i q_i\hat{\mathcal{D}}_i$. This result will help us to overcome the quantum ambiguity, since states of this form are compatible. It is sufficient for our purpose to take $\hat{\mathcal{D}}_{sub}(t_{split}) = |\Psi(t_{split})\rangle\langle\Psi(t_{split})|$ as being a projection on some pure state, since any $\hat{\mathcal{D}}_{sub}(t_{split})$ can be generated as a sum

of projections on its eigenvectors. To simplify the discussion, we assume the eigenvalues s_i of $\hat{s}$ to be non-degenerate, so that $\hat{r}_i = \hat{\Pi}_i = |s_i\rangle\langle s_i|$. Owing to the macroscopic size of A, the relative fluctuations of energy in the canonical states $\hat{\mathcal{R}}_i$ are small, and we may replace its canonical equilibrium states $\hat{\mathcal{R}}_i$ with broken invariance by microcanonical states $\hat{\mathcal{R}}_i^\mu$. The relative fluctuations of the pointer observable $\hat{A}$ around the value A_i are small in each state $\hat{\mathcal{R}}_i^\mu$, as well as the margin in which the energy lies for $\hat{\mathcal{R}}_i^\mu$. We note as $|A_i, \eta\rangle$ the kets of the Hilbert space of A that are involved, on the same footing, in $\hat{\mathcal{R}}_i^\mu$. As the spectrum is dense, the index η can take a very large number G of values, and $\hat{\mathcal{R}}_i^\mu$ has the form

$$\hat{\mathcal{R}}_i^\mu = \frac{1}{G} \sum_\eta |A_i, \eta\rangle\langle A_i, \eta|. \qquad (9)$$

We choose the initial time t_{split} for the dynamics of $\hat{\mathcal{D}}_{\text{sub}}$ sufficiently late so that $\hat{\mathcal{D}}(t_{\text{split}})$ has reached the form $\sum_i p_i \hat{\mathcal{D}}_i$ in which S and A are correlated through $\hat{\mathcal{D}}_i = |s_i\rangle\langle s_i| \otimes \hat{\mathcal{R}}_i^\mu$. (Due to the quantum ambiguity, we cannot yet assign $\hat{\mathcal{D}}_i$ to a subensemble of runs.) Hence the state $\hat{\mathcal{D}}(t_{\text{split}})$ involves only matrix elements within the Hilbert subspace of S+A spanned by the correlated kets $|s_i\rangle\otimes|A_i, \eta\rangle$. We can see that this property also holds for $\hat{\mathcal{D}}_{\text{sub}}(t_{\text{split}}) = |\Psi(t_{\text{split}})\rangle\langle\Psi(t_{\text{split}})|$ as a consequence of Eq (8): otherwise, the operator $\hat{\mathcal{D}}(t_{\text{split}}) - k\hat{\mathcal{D}}_{\text{sub}}(t_{\text{split}})$ for some $0 < k < 1$ could not equal the non-negative operator $(1-k)\hat{\mathcal{D}}_{\text{Csub}}$. The initial state allowed by this constraint has therefore the correlated form

$$|\Psi(t_{\text{split}})\rangle = \sum_{i,\eta} U(i,\eta)|s_i\rangle\otimes|A_i, \eta\rangle, \ \sum_{i,\eta}|U(i,\eta)|^2 = 1. \ (10)$$

The coupling $\hat{H}_{\text{SA}}$ has been switched off before t_{split}, and as in § 3, $\hat{H}_{\text{S}}$ is ineffective. Thus, the density operator $\hat{\mathcal{D}}_{\text{sub}}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ of S+A is governed for $t > t_{\text{split}}$ simply by the Hamiltonian $\hat{H}_{\text{A}}$. For suitable interactions within A, $\hat{H}_{\text{A}}$ induces among the kets $|A_i, \eta\rangle$ transitions which modify η, keeping unchanged the energy and the value of the pointer variable A_i (see [1] for examples). Then, for a large apparatus, an irreversible transformation takes place as in section 3, but over a shorter time scale τ_{sub} since no macroscopic energy transfer is involved. The mechanism is the same as the standard microcanonical relaxation [9], but several thermodynamic equilibrium states coexist here. Two types of irreversible processes simultaneously occur in the contributions of A to $\hat{\mathcal{D}}_{\text{sub}}(t)$. On the one hand, the contributions from $|A_i, \eta\rangle\langle A_j, \eta'|$ with $i \neq j$ fade out. On the other hand, for $i = j$, the state of A relaxes to equilibrium as in the standard relaxation towards a single microcanonical state (the terms with $\eta \neq \eta'$ disappear and the populations of the kets $|A_i, \eta\rangle$ equalise within each i, producing the factor $\hat{\mathcal{R}}_i^\mu$). Altogether [10], at the

final time $t_{\text{f}} > t_{\text{split}} + \tau_{\text{sub}}$, $\hat{\mathcal{D}}_{\text{sub}}$ reaches, within approximations standard in statistical mechanics and justified for a macroscopic apparatus, the expected form (5):

$$\hat{\mathcal{D}}_{\text{sub}}(t_{\text{f}}) \simeq \sum_i q_i \hat{r}_i \otimes \hat{\mathcal{R}}_i^\mu, \quad q_i = \sum_\eta |U(i,\eta)|^2. \quad (11)$$

Thus, provided some specific features of the apparatus produce the above dynamics, the form $\sum_i q_i \hat{\mathcal{D}}_i$ is reached for any mathematically admissible initial condition $\hat{\mathcal{D}}_{\text{sub}}(t_{\text{split}})$, in particular by the density operators $\hat{\mathcal{D}}_{\text{sub}}(t_{\text{f}})$ of S+A that describe the outcome of any physical subensemble $\mathcal{E}_{\text{sub}}$ of runs. The coefficients q_i of the various subensembles are related to one another by a *hierarchic structure*: If two subensembles $\mathcal{E}_{\text{sub}}^{(1)}$ and $\mathcal{E}_{\text{sub}}^{(2)}$ of $\mathcal{E}$, described by $\hat{\mathcal{D}}_{\text{sub}}^{(1)}$ and $\hat{\mathcal{D}}_{\text{sub}}^{(2)}$, and having $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ elements, respectively, merge into a subensemble $\mathcal{E}_{\text{sub}}$, the coefficients satisfy the standard addition law

$$\left[\mathcal{N}^{(1)} + \mathcal{N}^{(2)}\right] q_i = \mathcal{N}^{(1)} q_i^{(1)} + \mathcal{N}^{(2)} q_i^{(2)}. \qquad (12)$$

5. Emergence of classical probabilistic interpretation. As stated in § 1, quantum theory deals with an individual system only if it is embedded in a statistical ensemble, possibly a gedanken ensemble, but physically consistent. The state $\hat{\mathcal{D}}_i$ of S+A expected to describe the outcome (1) of a single run refers in fact to a specific subensemble $\mathcal{E}_i$. Within the statistical approach, physical subensembles $\mathcal{E}_{\text{sub}}$ (including $\mathcal{E}_i$) cannot a priori be identified, since they can be taken into consideration only after the time t_{split} when $\hat{\mathcal{D}}(t)$ has reached the form (4). However, suitable dynamics of the apparatus have allowed us to derive all the general properties that the physical subensembles must satisfy, to wit, the simple form (5) of their associated density operators and their hierarchic property (12).

To infer results about single runs, we note that this hierarchic structure of subensembles is just the same as that of ordinary probabilities, when these are interpreted as relative *frequencies* of occurrence of individual events [11]. Once we are ensured that the states of all possible subensembles have the form (5) and obey this structure, we are led to complement the principles of quantum mechanics with the following natural idea: We interpret the coefficients q_i associated with a subensemble $\mathcal{E}_{\text{sub}}$ as the proportion of runs, extracted from the full ensemble $\mathcal{E}$, having yielded the outcome A_i. This identification entails that $\mathcal{E}$ can be split, as in an ordinary probabilistic process, into subsets $\mathcal{E}_i$ of runs which provide the outcome A_i, each $\mathcal{E}_i$ containing Born's proportion p_i of runs. An individual run then belongs to one among the subsets $\mathcal{E}_i$, it can be assigned the state $\hat{\mathcal{D}}_i$, and *a solution of the quantum measurement problem is thereby found*.

Thus, a classical interpretation of some quantum objects emerges from a measurement process. The states defining the correspondence $\hat{O} \mapsto \text{tr}\hat{\mathcal{D}}\hat{O}$ have only a mathematical nature; the *c*-numbers $\langle\hat{O}\rangle$ present only a

formal analogy with ordinary expectation values. However, at the end of a measurement, we have just seen that the coefficients $p_i = \langle \hat{\Pi}_i \rangle = \text{tr}_S \hat{r}(0) \hat{\Pi}_i$ can be identified with ordinary probabilities in the frequency interpretation. Accordingly, $\langle \hat{s} \rangle = \sum_i p_i s_i$ and $A_i = \text{tr}_A \hat{\mathcal{R}}_i \hat{A}$ appear as ordinary statistical averages, and $\langle \hat{s}^2 \rangle - \langle \hat{s} \rangle^2$ as an ordinary variance; the correlation between $\hat{s}$ and $\hat{A}$ in $\hat{\mathcal{D}}(t_f)$ has a classical nature. Thus, in the same way as irreversibility emerges in statistical mechanics for macroscopic systems from the underlying reversible microscopic theory, a more remarkable type of emergence occurs at the end of a measurement process using a *macroscopic apparatus: Classical probabilities* emerge from quantum theory, although the underlying "quantum probabilities" were non-commutative and could not a priori be regarded as frequencies in the absence of an experimental context. Thus, certain properties of S, encoded statistically in the initial state $\hat{r}(0)$ of the ensemble $\mathcal{E}$, are selected by the measurement context and revealed by the outcomes of the individual runs of the process.

The occurrence of the factor $\hat{\mathcal{R}}_i$ in the state $\hat{\mathcal{D}}_i$ means that the runs are tagged by the value A_i of the pointer variable, which can be observed or registered at the macroscopic scale. By picking out the specific outcome A_i, one extracts from the whole ensemble $\mathcal{E}$ of runs the subensemble $\mathcal{E}_i$ that ends up in the state $\hat{\mathcal{D}}_i$. Two steps are thus necessary to go from the initial state $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}_i$. First, the evolution of the coupled system S+A for the full ensemble $\mathcal{E}$, governed by the Liouville–von Neumann equation (with approximations justified for large A), leads its state from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_f)$ expressed by (4); correlations are created, and information is transferred with some loss due to irreversibility. However, the subsequent reduction of the state from $\hat{\mathcal{D}}(t_f)$ to one of its components $\hat{\mathcal{D}}_i$ is not a consequence of some evolution, but the mere result of selecting A_i among the registered outcomes. This change of state, associated with the extraction from $\mathcal{E}$ of the subset $\mathcal{E}_i$, should be interpreted as an updating of information, in the same way as the change of an ordinary probability distribution, associated with some gain of information. A further step is involved in the *reduction of the state* of S from $\hat{r}(0)$ to $\hat{r}_i$, the disregard of A after selection of A_i.

Thus, the statistical formulation of quantum mechanics, though abstract and minimalist, which involves in fact no actual interpretation, is sufficient to explain all the features of ideal quantum measurements, and their classical interpretation, provided the apparatus is macroscopic and satisfies adequate dynamical properties. We thus advocate the usage of this formulation in teaching.

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[10] This type of relaxation also occurs in the dynamics of finite-temperature quantum phase transitions with spontaneously broken invariance, explaining within quantum theory why the order parameter may take a well-defined value in a single experiment. Hence this analogue of the measurement problem is solved along the same lines for macroscopic systems, but the problem will remain in phase transitions for finite systems.

[11] R. von Mises, *Probability, Statistics and Truth* (Macmillan, London, 1957).